

# Complete connections on fiber bundles

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## Abstract

Every fiber bundle admits a complete (Ehresmann) connection. After a first published incorrect proof, this has appeared in several other references, always relying in a more elaborated argument, that we have found to be incorrect too. We present here a definite proof for that theorem, together with illustrating examples, and some interesting consequences involving Riemannian submersions.

## 1 Introduction: A rather tricky exercise

An **(Ehresmann) connection** on a submersion  $p : E \rightarrow B$  is a smooth distribution  $H \subset TE$  that is complementary to the kernel of the differential, namely  $TE = H \oplus \ker dp$ . The distributions  $H$  and  $\ker dp$  are called **horizontal** and **vertical**, respectively, and a curve on  $E$  is called horizontal (resp. vertical) if its speed only takes values in  $H$  (resp.  $\ker dp$ ). Every submersion admits a connection: we can take for instance a Riemannian metric  $\eta^E$  on  $E$  and set  $H$  as the distribution orthogonal to the fibers.

Given  $p : E \rightarrow B$  a submersion and  $H \subset TE$  a connection, a smooth curve  $\gamma : I \rightarrow B$ ,  $t_0 \in I$ , locally defines a **horizontal lift**  $\tilde{\gamma}_e : J \rightarrow E$ ,  $t_0 \in J \subset I$ ,  $\tilde{\gamma}_e(t_0) = e$ , for  $e$  an arbitrary point in the fiber. This lift is unique if we require  $J$  to be maximal, and depends smoothly on  $e$ . The connection  $H$  is said to be **complete** if for every  $\gamma$  its horizontal lifts can be defined in the whole domain. In that case, a curve  $\gamma$  induces diffeomorphisms between the fibers by **parallel transport**. See e.g. [9] for further details. Some authors as [2, 3, 4] use other terminology and require an Ehresmann connection to be complete by definition.

A **fiber bundle**  $p : E \rightarrow B$  with fiber  $F$  is a submersion that admits an open cover  $\{U_i\}_i$  of  $B$ , and diffeomorphisms  $\phi_i : p^{-1}(U_i) \rightarrow U_i \times F$  such that  $\pi_1 \phi_i = p$ . Such a pair  $(U_i, \phi_i)$  is called a **local trivialization**. The purpose of this article is to show that, when  $B$  is connected, a submersion  $p : E \rightarrow B$  admits a complete connection if and only if it is a fiber bundle. One implication is easy: if  $H$  is a complete connection, working locally we can assume  $U_i = \{x \in \mathbb{R}^n : \|x\| < 1\}$  and define  $\phi_i : p^{-1}(U_i) \rightarrow U_i \times p^{-1}(0)$  by  $\phi_i(e) = (p(e), \tilde{\gamma}(0))$ , where  $\tilde{\gamma}$  is the lift of the segment  $\gamma(t) = tp(e)$  such that  $\tilde{\gamma}(0) = e$ . The converse, as we shall see, is definitely more challenging.

The result first appeared in [10, Cor 2.5] with a one-line incorrect proof that admits little analysis. Then it was posed as an exercise in [4, Ex VII.12], and presented as a theorem in [2, 6, 7, 8, 9]. These references all rely in an argument that P. Michor attributed in [7] to S. Halperin, one of the authors of the exercise. Unfortunately, we have found a mistake in this other proof, not perceived until now, and apparently insurmountable.

Given a submersion  $p : E \rightarrow B$ , if  $\eta^E$  is a metric on  $E$  that is complete and that makes  $p$  a Riemannian submersion, then it induces a complete connection  $H$ . This was the strategy behind the second proof, and it is in their construction of  $\eta^E$  that the issue appeared. Other failed construction of fibered complete metrics is offered in [10, Thm 3.6]. We will provide counter-examples to both constructions. Still, we manage to adapt our ideas to show that every fiber bundle do admit a fibered complete metric.

**Organization.** In section 2 we develop a criterion for completeness, based on the existence of enough flat sections, and use it to prove that every fiber bundle admits a complete connection. In section 3 we discuss the existence of fibered complete metrics, provide counter-examples to the constructions available in the literature, and present our solution.

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## 2 Our construction of complete connections

Given  $(U_i, \phi_i)$  a local trivialization of  $p : E \rightarrow B$ , there is an **induced connection** on  $p : p^{-1}(U_i) \rightarrow U_i$  defined by  $H_i = d\phi_i^{-1}(TU_i \times 0_F)$ , and is complete. The space of connections inherits a convex structure by identifying each connection  $H$  with the corresponding projection onto the vertical component. It is tempting then to construct a global complete connection, out of the ones induced by trivializations, by using a partition of 1. The problem is that, as stated in [4], complete connections are not closed under convex combinations.

**Example 1.** Let  $p : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the projection onto the first coordinate, and let  $H_1, H_2$  be the connections spanned by the following horizontal vector fields:

$$H_1 = \langle \partial_x + 2y^2 \sin^2(y) \partial_y \rangle \quad H_2 = \langle \partial_x + 2y^2 \cos^2(y) \partial_y \rangle$$

Both connections are complete. Note that the curves  $t \mapsto (t, k\pi)$ ,  $k \in \mathbb{Z}$ , integrate  $H_1$ , and because of them, any other horizontal lift of  $H_1$  is bounded and cannot go to  $\infty$ . The same argument applies to  $H_2$ . However, the averaged connection  $\frac{1}{2}(H_1 + H_2)$  is spanned by the horizontal vector field  $\partial_x + y^2 \partial_y$  and is not complete.

Our strategy to prove that every fiber bundle  $p : E \rightarrow B$  admits a complete connection is inspired by previous example. We will paste the connections induced by local trivializations by using a partition of 1, in a way so as to preserve enough local flat sections, that will bound any other horizontal lift of a curve. More precisely, given  $U \subset B$  an open, we say that a family of local sections  $\{\sigma_k : U \rightarrow E\}_k$  is **disconnecting** if the components of  $p^{-1}(U) \setminus \bigcup_i \sigma_i(U)$  have compact closure in  $E$ . We say that a local section  $\sigma$  is **tangent** to a connection  $H$  if  $d\sigma$  takes values in  $H$ .

**Lemma 2.** If every  $b \in B$  admits an open  $U \subset B$  and a disconnecting family of sections  $\{\sigma_k : U \rightarrow E\}_k$  that are tangent to a connection  $H$ , then  $H$  has to be complete.

*Proof.* Let  $U \subset B$  be an open that admits a disconnecting family of sections  $\{\sigma_k : U \rightarrow E\}_k$  that are tangent to  $H$ . Given a curve  $\gamma : [t_0, t_1] \rightarrow U \subset B$ ,  $\gamma(t_0) = b$ , we will show that it can be lifted with arbitrary initial point. This will be enough for  $B$  can be covered with opens of this type. If we lift the initial point  $b$  to a point  $\sigma_k(b)$ , then we can lift the whole  $\gamma$  by using  $\sigma_k$ . If we lift  $b$  to a point not in  $\bigcup_k \sigma_k(U)$ , then it will remain within the same component of  $p^{-1}(U) \setminus \bigcup_k \sigma_k(U)$ , that is contained in a compact, and therefore we can easily extend the lift to the whole domain.  $\square$

From here on, let us fix a proper positive function  $h : F \rightarrow \mathbb{R}$ , we call it the **height** function, it somehow controls the distance to  $\infty$ . We can take for instance  $h(x) = \sum_n n\lambda_n$  where  $\{\lambda_n\}_n$  is a locally finite countable partition of 1 by functions of compact support. Given  $(U_i, \phi_i)$  a local trivialization, we define the **tube** of radius  $n$  over  $U_i$  as the set  $T_i(n) = \phi_i^{-1}(U_i \times h^{-1}(n))$ . Note that if  $N_i \subset \mathbb{N}$  is infinite, then  $\{\sigma_f : b \mapsto \phi_i^{-1}(b, f)\}_{f \in h^{-1}(N_i)}$  is a disconnecting family of sections that are tangent to the induced connection over  $U_i$ , for they form an infinite union of tubes of unbounded radiuses.

**Theorem 3.** Every fiber bundle  $p : E \rightarrow B$  admits a complete connection  $H$ .

*Proof.* The strategy will be to take a nice covering of  $B$  by trivializing opens  $(U_i, \phi_i)$ , take an infinite union of tubes  $T_i = \bigcup_{n \in N_i} T_i(n)$  over each  $U_i$  in such a way that their closures do not intersect, and finally construct by using a partition of 1 a connection  $H$  that over  $T_i$  agrees with the induced connection  $H_i$  by a local trivialization.

To start with, let  $\{U_i : i \in \mathbb{N}\}$  be a countable open cover of  $B$  such that (i) it is locally finite, (ii) each open  $U_i$  has compact closure, and (iii) the closure of each open is contained in a trivialization  $(V_i, \phi_i)$ . The construction of such a cover is rather standard.

Next, we will define inductively the radius of our tubes, starting with a tube over  $U_1$ , then another over  $U_2$ , and so on, until we have constructed one tube over each open  $U_i$ . After that, we will construct a second tube over  $U_1$ , then a second tube over  $U_2$ , and so on. This process will end up providing infinitely many tubes over each open set  $U_i$ .

At the moment of constructing the  $j$ -th tube over  $U_i$ , the open  $p^{-1}(U_i)$  can only intersect finitely many previously built tubes. The closure of this intersection will be compact, and the function  $h\pi_2\phi_i$  will attain a maximum there. We can pick the radius of the new tube as the minimum integer bigger than that maximum.

Finally, set  $\{\lambda_i\}_i$  a partition of 1 subordinated to  $W_i = p^{-1}(U_i) \setminus \bigcup_{i \neq j} \bar{T}_j$ , and set  $H = \sum_i \lambda_i H_i$ , where  $H_i$  is the connection induced by  $(U_i, \phi_i)$ . If  $x \in T_i$  and  $j \neq i$  then  $\lambda_j(x) = 0$ , and therefore  $H = H_i$  over  $T_i$ . It follows that  $\{\sigma_f : f \in h^{-1}(N_i)\}$  is a disconnecting family of sections over  $U_i$  that are tangent to  $H$ , then by our criterion  $H$  has to be complete.  $\square$

**Remark 4.** Let us mention two particular cases on which the problem admits a simple solution. If the fiber  $F$  is compact then the map  $p : E \rightarrow B$  is proper and, therefore, any lift of a curve can be extended to the whole domain, and any connection is complete. This was already noted in [3]. Other well-known case is when  $p : E \rightarrow B$  is a principal bundle with Lie group  $G$ . In that case, if  $H$  is constructed so as to be  $G$ -invariant, then the several local lifts of a curve can be translated by  $G$  so as to agree in the intersections and define a global lift. These arguments, however, are of little help when addressing the general case.

### 3 Fibred complete Riemannian metrics

A **Riemannian submersion**  $p : E \rightarrow B$  is a submersion between Riemannian manifolds such that the maps  $dp_e : T_e F^\perp \cong T_{p(e)} B$  is an isometry for all  $e \in E$ . As in [1], we will say that a metric on  $E$  is **fibred** if the compositions  $T_e F^\perp \cong T_b B \cong T_{e'} F^\perp$  are isometries for every pair  $e, e'$  of points lying on the same fiber. A fibred metric on  $E$  clearly induces a metric on  $B$  that makes the submersion Riemannian.

A fundamental feature of Riemannian submersions is that the horizontal lifts of geodesics are geodesics. It follows that the exponential maps induce a commutative square as below,

where the dash arrows are only defined around the zero section  $0_F$  and the point  $0_b$ .

$$\begin{array}{ccccc}
 T_b B \times F & \cong & TF^\perp & \xrightarrow{\text{exp}} & E \\
 & \searrow \pi_1 & \downarrow dp & & \downarrow p \\
 & & T_b B & \xrightarrow{\text{exp}} & B
 \end{array}$$

If we happen to have a metric  $\eta^E$  that is both fibered and complete, since  $TF^\perp$  is trivial and  $dp$  identifies with the projection, we can get a local trivialization of  $p$ , as explained in the theorem below. It is easy to see that every manifold admits a complete metric, and that every submersion admits a fibered metric, but imposing both conditions simultaneously is a more delicate issue, and in fact not always possible.

The construction of complete connections available in [2, 6, 7, 8, 9] is based on a fibered complete metric, constructed as a convex combination of local fibered metrics. The problem is that fibered metrics are not closed under convex combinations, as simple counterexamples show, see for instance [1, Ex 2.1.3]. One can export from the dual bundle a convex structure on the set of fibered metrics, or define other ad hoc convex structures on this set, but then the required bounds used in the argument no longer hold.

Next we adapt our ideas to construct fibered complete metrics on any fiber bundle.

**Theorem 5.** Given  $p : E \rightarrow B$  a submersion,  $B$  connected, the following are equivalent:

- (i)  $p$  is locally trivial;
- (ii)  $p$  admits a complete connection  $H$ ;
- (iii) there is a metric  $\eta^E$  on  $E$  that is both fibered and complete.

*Proof.* We have already shown (i)  $\Leftrightarrow$  (ii). To show (iii)  $\Rightarrow$  (i), let  $0 \in U \subset T_b B$  be an open over which the exponential of the induced metric  $\eta^B$  is an open embedding. It follows from [1, Prop. 5.2.2] that the exponential of  $\eta^E$  restricted to  $\tilde{U} = dp^{-1}(U) \cap TF^\perp$  is also an open embedding and hence it defines a trivialization of  $p$  around  $b$  (see also [5]).

Let us prove (i)  $\Rightarrow$  (iii). We construct a fibered metric on  $E$  in the similar fashion we have constructed  $H$  on the proof of the theorem. Set  $\eta^B$  a complete metric on the base, and  $\eta^F$  a complete metric on the fiber. Define a family of tubes inductively as in that other proof, but now taking **thick tubes**  $\tilde{T}_i(n) = \phi^{-1}(U_i \times h^{-1}(n, n + l_n))$ , where  $l_n$  is so as to insure that the distance between  $h^{-1}(-\infty, n)$  and  $h^{-1}(n + l_n, \infty)$  is at least 1. Call  $\tilde{T}_i = \bigcup_{n \in \mathbb{N}_i} \tilde{T}_i(n)$ . We get a global metric by the convex combination  $\eta^E = \sum_i \lambda_i \phi_i^*(\eta^B|_{U_i} \times \eta^F)$ , where  $\lambda_i$  is a partition of 1 subordinated to  $W_i = p^{-1}(U_i) \setminus \bigcup_{i \neq j} \tilde{T}_j$ . Next we see that it is complete.

Let  $\gamma : [t_0, t_1] \rightarrow E$  be a unit-speed geodesic,  $t_0, t_1 \in \mathbb{R}$ . The projection  $p\gamma$  has speed bounded by one, then it is Lipschitz, and since  $B$  is a complete metric space (Hopf-Rinow), there exists  $b = \lim_n p\gamma(t_1 - \frac{1}{n}) = \lim_{t \rightarrow t_1} p\gamma(t)$ . Let  $(U_i, \phi_i)$  be one of the trivializations around  $b$  used to construct  $\eta^E$ . If  $\gamma$  is included in some compact  $K \subset E$  then there exists  $e = \lim_{t \rightarrow t_1} \gamma(t)$  and the geodesic can be extended easily. If there is no such  $K$ , then the function  $h\pi_2\phi_i$  cannot be bounded over the image of  $\gamma$ , and  $\gamma$  has to cross infinitely many thick tubes  $\tilde{T}_i(n)$  on finite time. Since  $\eta^E$  and  $\phi_i^*(\eta^B|_{U_i} \times \eta^F)$  agree over these tubes,  $\gamma$  will need at least time 1 to cross each of them, which leads us to a contradiction.  $\square$

To finish, we deal with the construction in [10, Thm 3.6], as promised in the introduction. Starting with a complete metric on  $E$ , they build a new metric by preserving the induced

connection and the vertical component, and replacing the horizontal component by the lift of a complete metric on  $B$ . Contrary to what is asserted, the resulting metric need not to be complete, even if the induced connection is complete, as next example shows.

**Example 6.** Let  $a : \mathbb{R} \rightarrow [0, 1]$  be smooth and such that  $a(0) = 1$  and  $a(x) = 0$  if  $|x| \geq 1$ . Let  $b : (-1, 1) \rightarrow \mathbb{R}$  be smooth increasing and such that  $b'(0) = 0$  and  $\lim_{x \rightarrow \pm 1} b(x) = \pm\infty$ . Define  $\phi_0 : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  by  $\phi_0(x, y) = a(x - b(y - 4) - 4)$  if  $3 < y < 5$  and 0 otherwise. Construct now a sequence  $\phi_k : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  by  $\phi_k(x, y) = \phi(2^k x, 2^k y)$ . The supports of the  $\phi_k$  are disjoint and hence  $\phi = \sum_{k \geq 0} \phi_k$  is well-defined and smooth. The graph of  $\phi$ , that we denote  $E$ , can be thought of as a chain of hills of height 1 approaching the  $x$ -axis. Let  $p : E \rightarrow \mathbb{R}$  be the second projection. It is easy to see that  $E$ , with the induced metric from  $\mathbb{R}^3$ , is complete. Moreover, the induced connection  $H$  is complete, for the family  $\{\sigma_k : t \mapsto (t, \frac{4}{2^k})\}_{k \in \mathbb{Z}}$  is disconnecting, and since  $b'(0) = 0$ , tangent to  $H$ . Now construct a new metric  $\eta^E$  on  $E$  as in [10], by preserving the induced connection and vertical component, and lifting to the horizontal distribution the standard metric on  $\mathbb{R}$ . Then  $\eta^E$  is fibered over a complete metric and its vertical component comes from a complete metric, but is not complete. In fact, if  $c : (-5, 0) \rightarrow \mathbb{R}$  is smooth, decreasing and such that  $c(x) = \frac{4}{2^k}$  for  $-\frac{5}{2^k} \leq x \leq -\frac{3}{2^k}$ , then the curve  $t \mapsto (t, c(t))$  has finite length and cannot be extended to 0.

## References

- [1] M. del Hoyo, R. Fernandes; Riemannian metrics on Lie groupoids; To appear in J. reine angew. Math., 2015.
- [2] W. de Melo; Topologia das Variedades; draft available at <http://w3.impa.br/~demelo/>
- [3] C. Ehresmann; Les connexions infinitésimales dans un espace fibré différentiable; Colloque de Topologie, Bruxelles, 1950, pp. 29-55.
- [4] W. Greub, S. Halperin, R. Vanstone; Connections, Curvature, and Cohomology I; Academic Press, New York and London, 1972.
- [5] R. Hermann; A sufficient condition that a map of Riemannian manifolds be a fiber bundle; Proc. Amer. Math. Soc. 11 (1960), 236-242.
- [6] I. Kolář, P. Michor, J. Slovák; Natural operations in differential geometry; Springer-Verlag, Berlin, Heidelberg, New York, 1993.
- [7] P. Michor; Gauge theory for diffeomorphism groups; Proc. of the Conf. on Diff. Geom. Methods in Theoretical Physics, Como 1987, Kluwer, Dordrecht, 1988, p. 345-371.
- [8] P. Michor; Gauge theory for fiber bundles; Monographs and Textbooks in Physical Sciences 19, Bibliopolis, Napoli, 1991.
- [9] P. Michor; Topics in Differential Geometry; Graduate Studies in Mathematics 93, American Mathematical Society, 2008.
- [10] J. Wolf; Differentiable fibre spaces and mappings compatible with riemannian metrics; Michigan Mathematical Journal, vol. 11 (1964), pp. 65-70.

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